The University of Georgia

Mathematics Education Program
J. Wilson, EMAT 6690

Minimal triangle via internal point P in an angle

By

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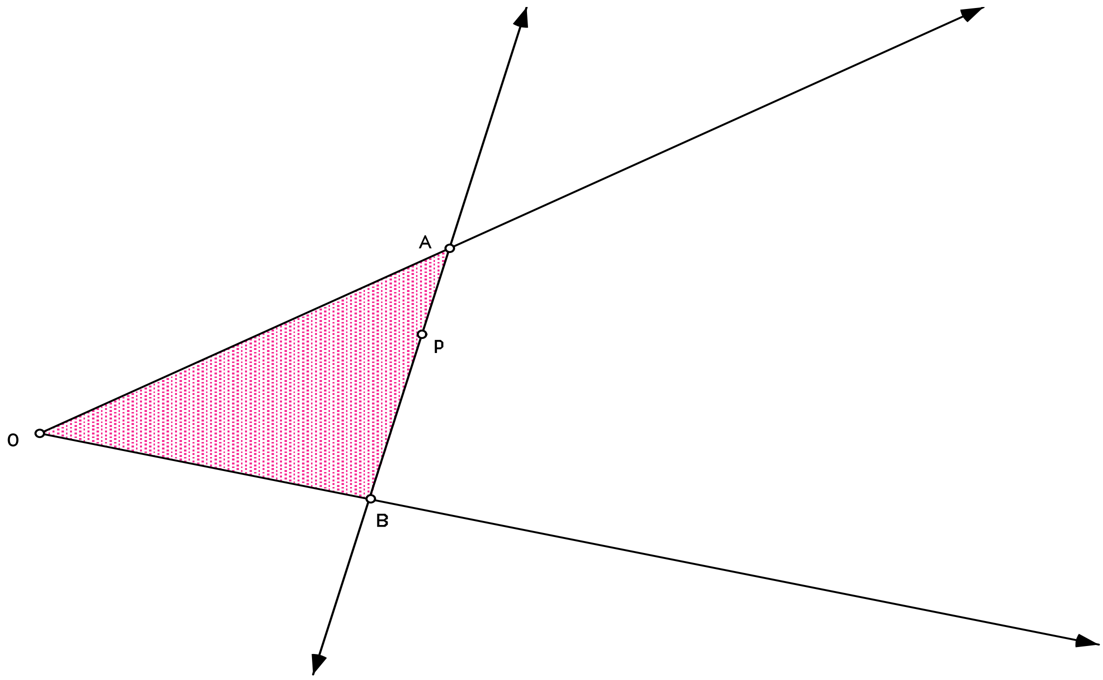
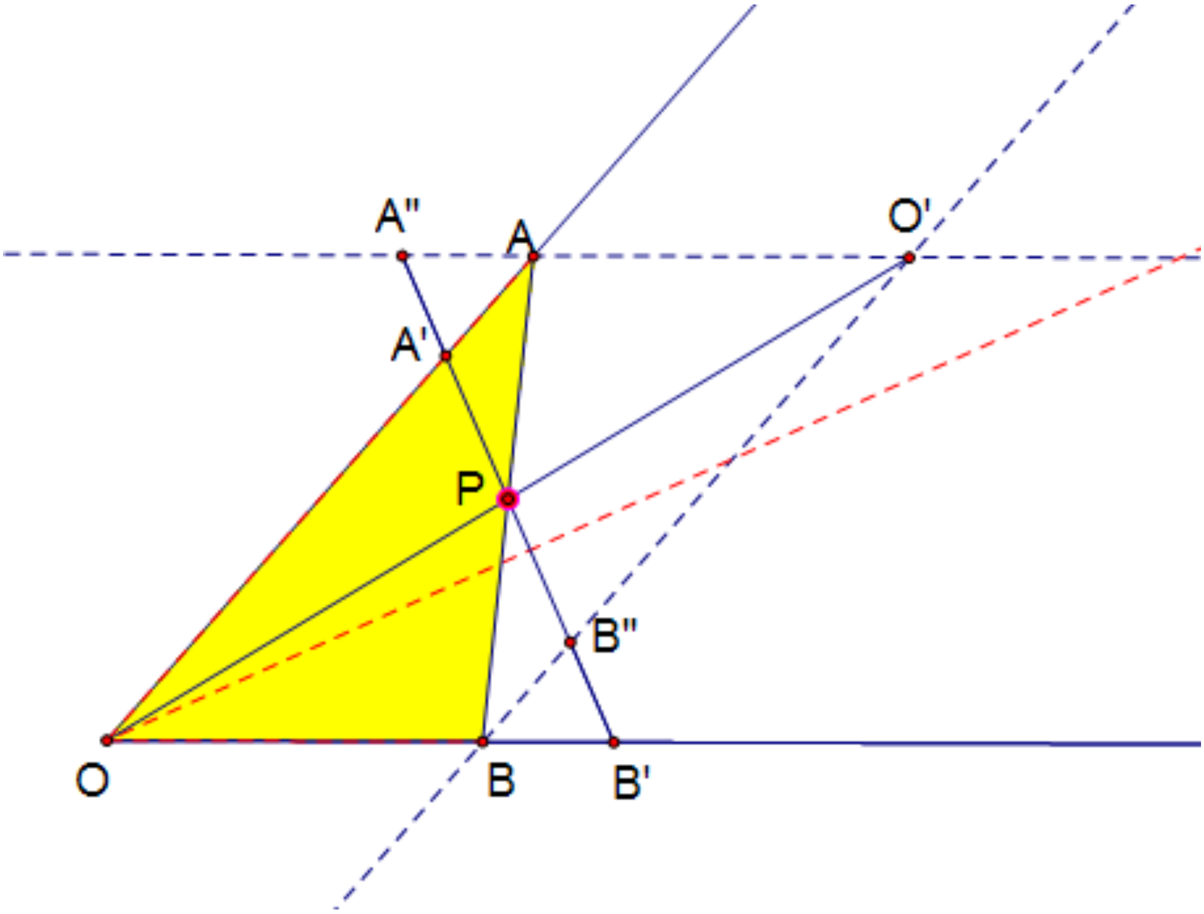
Q. Given an angle in a plane with vertex O and a point P in the interior of the angle.
Take a line through P intersecting the sides of the angle at points A and B.



Exploration by GSP

To find out minimal area of the triangle AOB, we can animate a point A.
Given any angle AOB, intuitively, there are two cases of p points, which lie on the angle bisector and not on the angle bisector.

Case 1) P is not on the angle bisector (The red dashed ray is the angle bisector.)

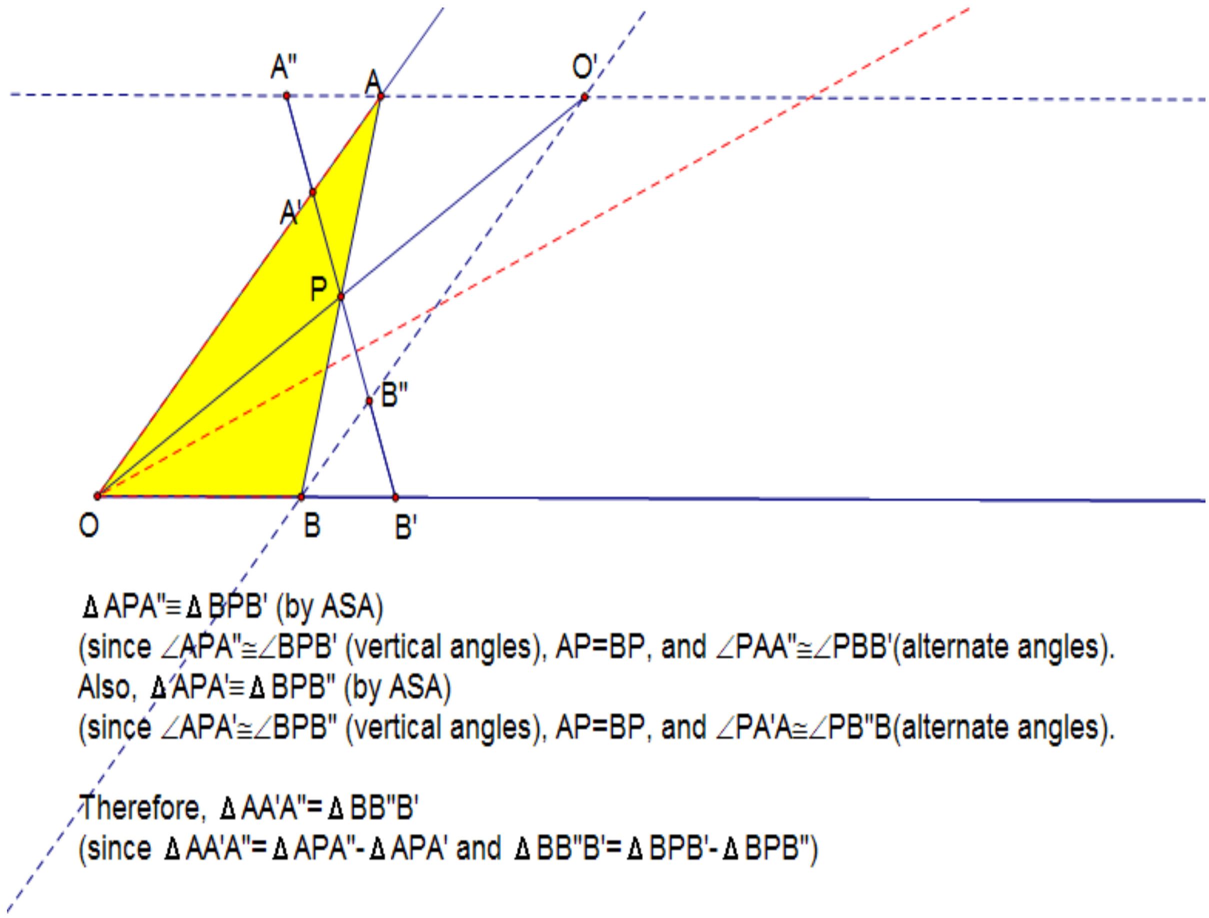


How to Construct

1. Make a point O’ on the ray OP such that OO’ = 2OP.
2. Construct lines parallel to the sides of the angle through O’.
3. Let A and B be the points of intersection of the pairs of the lines.

Obviously, the parallel lines make a parallelogram OAO’B with P the midpoint of the diagonal OO'. It is then also the midpoint of the diagonal AB so that AB passes through P. Since two diagonals of parallelogram intersect at one point.

Let A'B' be another line through P. A'B' intersects AO' in A" and BO' in B".



Now, Area (OA'B') = Area (O'A"B"),

Area(OA'B') = [Area(OAO'B) + Area(AA'A") + Area(BB'B")] / 2

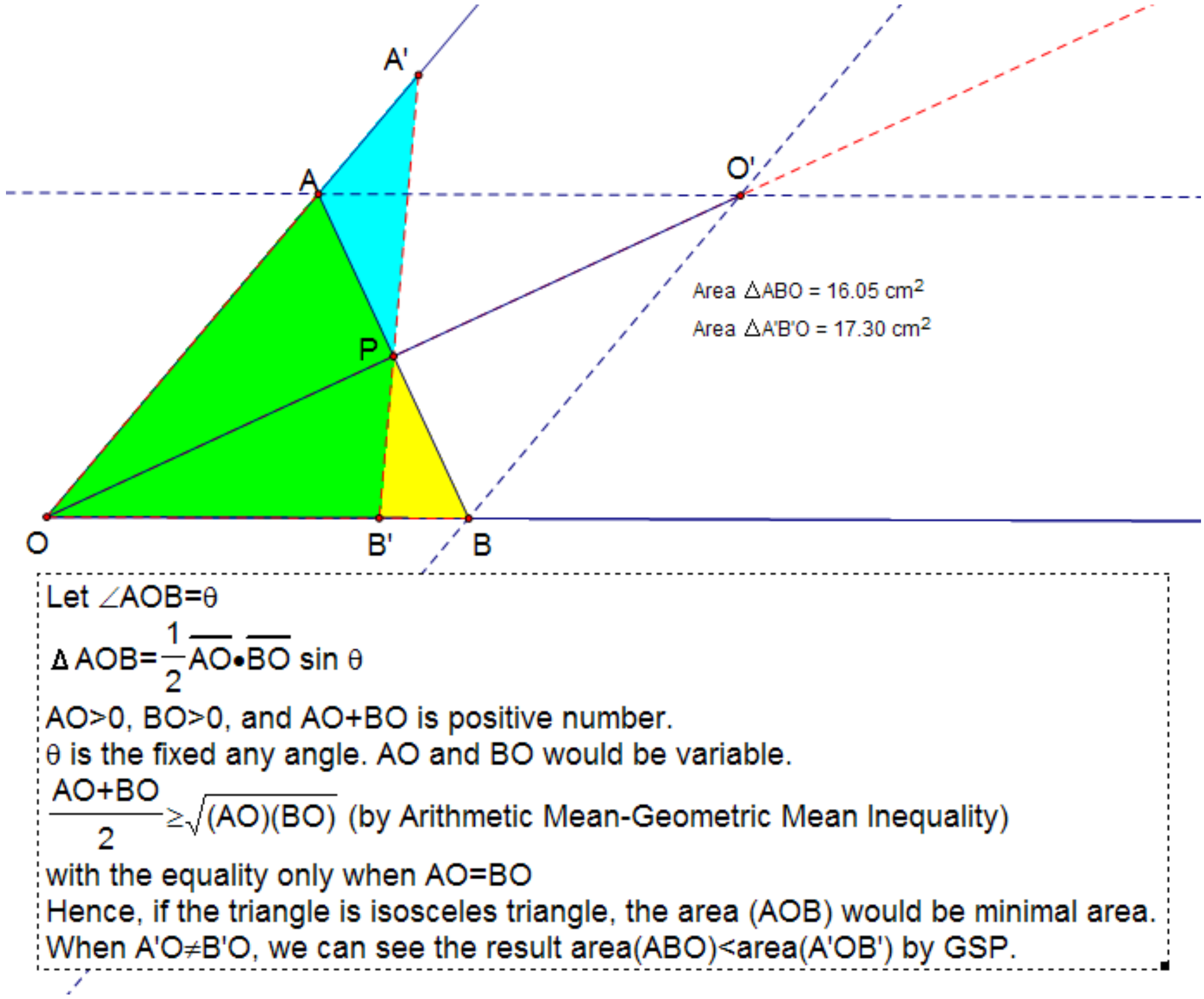
= Area(OAB) + {Area(AA'A") + Area(BB'B")} / 2

≥ Area(OAB),

With the equality only when A = A' = A" and B = B' = B"

Thus, the triangle area OAB would be minimal area if A = A' = A" and B = B' = B"

Case 2) P lies in the angle bisector.



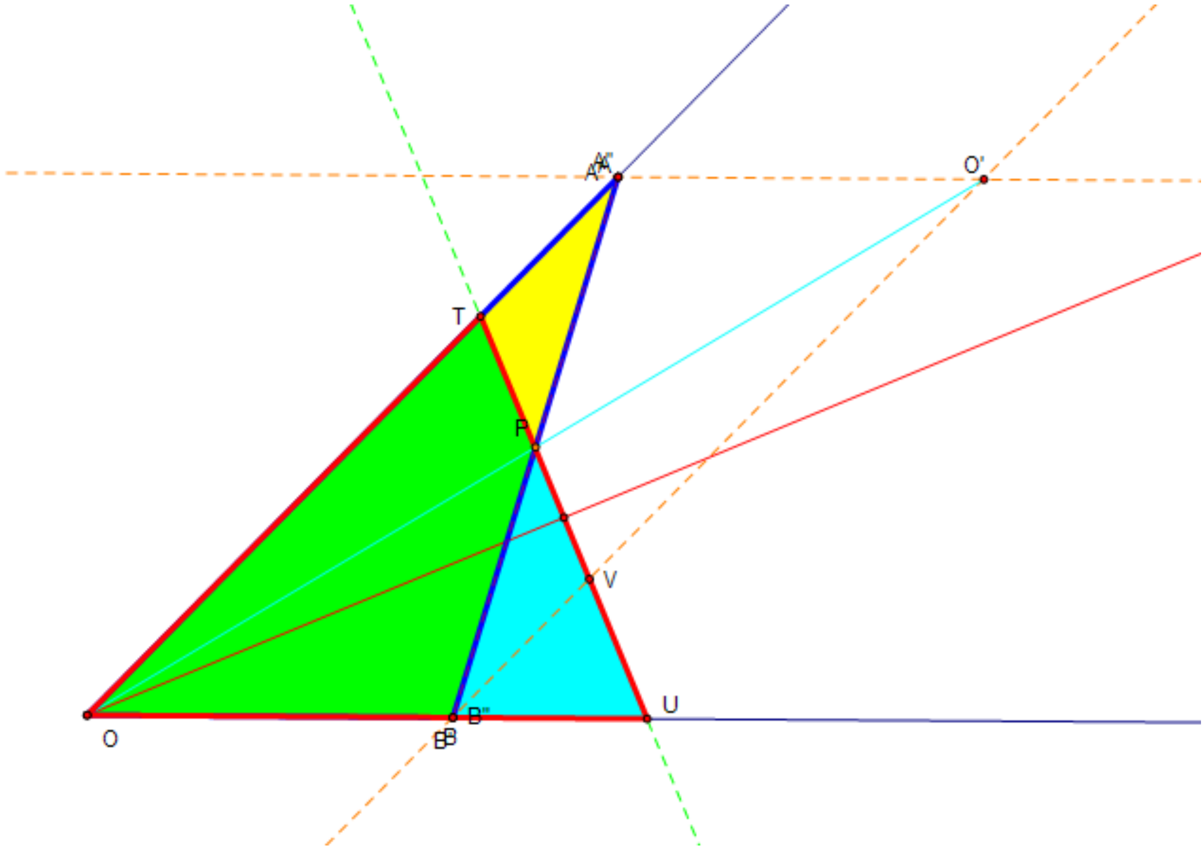
Investigation

The area of triangle AOB would be smallest (nearly zero) as a point P approaches either a point O or sides of the angle. When a point P goes to a point O, the triangle would be close to one point. Also, when a point P approaches sides of the angle, the triangle would be close to the same segments. The line generating the minimal area is not unique.

Geometric backgrounds as follows.

- Properties of parallelogram: 1) the opposite sides are congruent and parallel,
  2) the diagonals intersect at one point and bisect the area.
- Triangle congruence such as SSS, SAS, and ASA.
- The equality of vertical angles and alternate angles when two lines are parallel.
- Triangle area formula: (1/2)(side)(side)sin(theta).
- Arithmetic Mean-Geometric Mean Inequality
- How to construct the angle bisector

Comparison



The green dashed line is perpendicular to the angle bisector (thin red ray).
If I restrict that P moves between T and U, (case 1) would be less than (case 2).

Case 1) The thick blue outline triangle represents minimal area when P is not on the angle bisector.
Case 2) The thick red outline triangle represents minimal area when P lies on the angle bisector.
Because of ATP=BVP, quadrilateral TOBV is smaller than isosceles triangle TOU.
Thus, the isosceles triangle is not the one with minimal area.

Reference Link
http://jwilson.coe.uga.edu/emt725/MinTri/MinTri.html